

On the supersymmetry of the Dirac-Kepler problem plus a Coulomb-type scalar potential in $D + 1$ dimensions and the generalized Lippmann-Johnson operator

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We study the Dirac-Kepler problem plus a Coulomb-type scalar potential by generalizing the Lippmann-Johnson operator to D spatial dimensions. From this operator, we construct the supersymmetric generators to obtain the energy spectrum for discrete excited eigenstates and the radial spinor for the SUSY ground state.

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I. INTRODUCTION

The Dirac-Kepler problem in $D+1$ dimensions has been treated in several ways: by power series¹, radial SUSY QM², supersymmetry generated by the Lippmann-Johnson operator³ and an $su(1,1)$ approach⁴. On the other hand, the Dirac radial equations with vector and scalar Coulomb-type potentials were studied by series power⁵, SUSY QM⁶ and intertwining operators⁷.

Joseph was the first who studied the Dirac equation with vector potential in D spatial dimensions⁸. Also, the Lippmann-Johnson operator was introduced to generate the supersymmetric charges⁹. With Coulomb-type scalar and vector potentials, the problem was solved in analytical way by reducing the radial Dirac equations to the differential equations satisfied by confluent hypergeometric functions^{10,11}. The energy spectrum and the SUSY ground state of this problem were found by an $su(1,1)$ algebraic approach¹⁴. However, to our knowledge the Lippmann-Johnson operator in general dimensions with Coulomb-type scalar and vector potentials has not been reported. Therefore, the supersymmetry generated from this constant of motion remains untreated.

The purpose of this paper is to construct the supersymmetry charges for the Dirac-Kepler problem plus a Coulomb scalar potential from the generalized Lippmann-Johnson operator in D dimensions. The fact that one of the supercharges annihilates the SUSY ground state leads us to find the energy spectrum for discrete excited states and to obtain the radial differential equations for the SUSY ground state. By performing a similarity transformation to the radial Lippmann-Johnson operator we find the SUSY ground state. Also, we show that the radial part of the Lippmann-Johnson operator is reduced to that reported in Ref. 7 for the three-dimensional space and obtained by intertwining considerations.

II. THE RELATIVISTIC DIRAC EQUATION IN $D+1$ DIMENSIONS AND THE GENERALIZED LIPPMANN-JOHNSON OPERATOR

The Dirac equation in $D+1$ dimensions for a central field can be written with ($\hbar = c = 1$) as⁵

$$H\Psi \equiv \left\{ \gamma^0 \gamma^j p^j + \gamma^0 (m + V_s(r)) + V_v(r) \right\} \Psi = i \frac{\partial \Psi}{\partial t}, \quad j = 1, 2, \dots, D \quad (1)$$

where summation over repeated index is assumed, m is the mass of the particle, V_s and V_v are the spherically symmetric scalar and vector potentials, respectively, and the Dirac matrices in D dimensions, γ^j , satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with

$$\eta^{\mu\nu} = \begin{cases} \delta^{\mu\nu} & \text{if } \mu = 0, \\ -\delta^{\mu\nu} & \text{if } \mu \neq 0. \end{cases} \quad (2)$$

In D spatial dimensions, the orbital angular momentum operator L_{ab} and the total angular momentum J_{ab} are defined as

$$L_{ab} = ix_a \partial_b - ix_b \partial_a \quad (3)$$

and

$$J_{ab} = L_{ab} + \frac{i}{2} \gamma^a \gamma^b, \quad (4)$$

respectively. In the case of spherically symmetric potentials, the total angular momentum operator and the spin-orbit operator

$$K_D = -\gamma^0 \left\{ \frac{i}{2} \sum_{a \neq b} \gamma^a \gamma^b L_{ab} + \frac{1}{2} (D-1) \right\} \quad (5)$$

commute with the Dirac Hamiltonian. For a given total angular momentum j , the eigenvalues of K_D are $\kappa_D = \pm (j + 1/2)$, where the minus sign is for aligned spin $j = \ell + \frac{1}{2}$, and the plus sign is for unaligned spin $\ell - \frac{1}{2}$ ¹¹.

We find that if the scalar and vector potentials are given by $V_s = \frac{\alpha_s}{r}$ and $V_v = \frac{\alpha_v}{r}$ then, the matrix hermitian operator

$$B = -iK_D \gamma^{D+1} (H - \gamma^0 m) + \gamma^{D+1} \gamma^0 \gamma^i \frac{x^i}{r} (\alpha_v m + \alpha_s H) \quad (6)$$

is a constant of motion (see appendix A), where the pseudoscalar γ^{D+1} is reduced to the matrix γ^5 in $(3+1)$ dimensions and satisfies $(\gamma^{D+1})^\dagger = \gamma^{D+1}$, $(\gamma^{D+1})^2 = 1$ and $\{\gamma^{D+1}, \gamma^\mu\} = 0$. Also, it can be shown that this operator anticommutes with the Dirac operator K_D . Therefore, the operator B satisfies

$$B \Psi_{\kappa_D} = -b \Psi_{-\kappa_D}, \quad (7)$$

where b is an undetermined constant. In fact, B is the generalization of the Lippmann-Johnson operator¹² and is reduced to that given in Ref. 7 for $D = 3$, and to that for D dimensions in absence of the scalar potential V_s reported in Ref. 9.

In this way, for odd or even dimensions, we write the eigenstates of equation (1) as

$$\Psi_{\kappa_D} = r^{-\frac{D-1}{2}} \begin{pmatrix} G_{\kappa_D}^{(1)}(r) \chi_{\kappa_D}^{\mu}(\Omega_D) \\ iG_{\kappa_D}^{(2)}(r) \chi_{-\kappa_D}^{\mu}(\Omega_D) \end{pmatrix} e^{-iEt}, \quad (8)$$

being $G_{\kappa_D}^{(1)}$ and $G_{\kappa_D}^{(2)}(r)$ the radial functions, and $\chi_{\kappa_D}^{\mu}(\Omega_D)$ the hyperspherical harmonic functions coupled with the angular momentum j^{11} . We consider that

$$\gamma^{D+1} \gamma^0 \gamma^i \frac{x^i}{r} \Psi_{\kappa_D} = -\Psi_{-\kappa_D}, \quad (9)$$

which is the generalization of the three-dimensional equation $(\vec{\sigma} \cdot \hat{r}) \Psi_{\kappa} = -\Psi_{-\kappa}$ ^{3,15}. By defining the spinors

$$\Theta_{\kappa_D} = \begin{pmatrix} -G_{\kappa_D}^{(2)} \chi_{\kappa_D} \\ iG_{\kappa_D}^{(1)} \chi_{-\kappa_D} \end{pmatrix} e^{-iEt}, \quad \Phi_{\kappa_D} = \begin{pmatrix} G_{\kappa_D}^{(1)} \chi_{\kappa_D} \\ iG_{\kappa_D}^{(2)} \chi_{-\kappa_D} \end{pmatrix} e^{-iEt} \quad (10)$$

and from the results of Appendix B, the explicit form of the operator B acting on a general eigenstate of the Dirac Hamiltonian is

$$B\Psi_{\kappa_D} = -r^{-\frac{D-1}{2}} \left\{ (\alpha_s \partial_r - \kappa_D V_v) \Theta_{\kappa_D} + \left(\kappa_D \left(\partial_r + \frac{\kappa_D}{r} \gamma^0 \right) + m\alpha_v \right) \Phi_{\kappa_D} + \alpha_s \gamma^0 (m + V_s + \gamma^0 V_v) \Phi_{\kappa_D} \right\}. \quad (11)$$

III. SUSY QM AND THE ENERGY SPECTRUM

Based on Refs. 3,9,13, we define the supersymmetric generator

$$Q = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \quad (12)$$

which satisfies $\{Q, Q\} = 0$ and

$$\mathcal{H} \equiv \{Q, Q^{\dagger}\} = \begin{pmatrix} B^2 & 0 \\ 0 & B^2 \end{pmatrix}, \quad (13)$$

with \mathcal{H} the supersymmetric Hamiltonian.

In order to obtain the energy spectrum for the Dirac Hamiltonian H , we consider the results given in Appendix C, from which

$$B^2 = (\alpha_v m + \alpha_s H)^2 + K_D^2 (H^2 - m^2). \quad (14)$$

By considering equation (7), we have the eigenvalue equation $B^2\Psi_{\kappa_D} = b^2\Psi_{\kappa_D}$. Thus

$$b^2 = (\alpha_v m + \alpha_s E)^2 + \kappa_D^2 (E^2 - m^2). \quad (15)$$

Since the supersymmetric ground state, Ψ_{SUSY}^0 must satisfy the condition

$$\mathcal{H}\Psi_{SUSY}^0 = 0, \quad (16)$$

it follows that the ground state energy eigenvalue, E_0 , is obtained from equation (15) by setting $b = 0$. In this way

$$E_0 = m \left\{ -\frac{\alpha_s \alpha_v}{\alpha_v^2 + \gamma^2} \pm \sqrt{\left(\frac{\alpha_s \alpha_v}{\alpha_v^2 + \gamma^2}\right)^2 - \left(\frac{\alpha_s^2 - \gamma^2}{\alpha_v^2 + \gamma^2}\right)} \right\}, \quad (17)$$

where $\gamma^2 = \kappa_D^2 + \alpha_s^2 - \alpha_v^2$. For the excited states of the Hamiltonian H , we perform the change $\gamma \rightarrow \gamma + n$, where $n = 0, 1, 2, 3, \dots$ is the radial quantum number. Then

$$\frac{E_n}{m} = -\frac{\alpha_s \alpha_v}{\alpha_v^2 + (\gamma + n)^2} \pm \sqrt{\left(\frac{\alpha_s \alpha_v}{\alpha_v^2 + (\gamma + n)^2}\right)^2 - \left(\frac{\alpha_s^2 - (\gamma + n)^2}{\alpha_v^2 + (\gamma + n)^2}\right)}, \quad (18)$$

which is in accordance to that obtained from an analytical^{5,11} or $su(1, 1)$ algebraic approach¹⁴.

The eigenstates Ψ_{κ_D} and $\Psi_{-\kappa_D}$ are transformed into each other by the operator B (equation (7)) and both are eigenfunctions of the operator B^2 with the same eigenvalue. Therefore, the supersymmetric eigenstates can be written as

$$\Psi_{SUSY} = \begin{pmatrix} \Psi_{\kappa_D} \\ \Psi_{-\kappa_D} \end{pmatrix}. \quad (19)$$

Considering equations (13) and (16), the components of the supersymmetric ground state must satisfy

$$B\Psi_{0 \pm \kappa_D} = 0. \quad (20)$$

In order to solve this equation for $+\kappa_D$ (the solution for the other sign can be obtained by equation (7)) we consider the expression (B3), from which

$$\begin{aligned} B\Psi_{\kappa_D} = & -r^{-\frac{D-1}{2}} e^{-iEt} \left\{ (\alpha_s \partial_r - \kappa_D V_v) \begin{pmatrix} -G_{\kappa_D}^{(2)} \chi_{\kappa_D} \\ iG_{\kappa_D}^{(1)} \chi_{-\kappa_D} \end{pmatrix} + (\kappa_D \partial_r + m\alpha_v + \alpha_s V_v) \begin{pmatrix} G_{\kappa_D}^{(1)} \chi_{\kappa_D} \\ iG_{\kappa_D}^{(2)} \chi_{-\kappa_D} \end{pmatrix} \right. \\ & \left. + \left(\frac{\kappa_D^2}{r} + \alpha_s (m + V_s) \right) \begin{pmatrix} G_{\kappa_D}^{(1)} \chi_{\kappa_D} \\ -iG_{\kappa_D}^{(2)} \chi_{-\kappa_D} \end{pmatrix} \right\}. \end{aligned} \quad (21)$$

Thus, the radial components of the SUSY ground state ($G_{0\kappa_D}^{(1)}$ and $G_{0\kappa_D}^{(2)}$) satisfy

$$L_D \begin{pmatrix} G_{0\kappa_D}^{(1)} \\ G_{0\kappa_D}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (22)$$

where

$$L_D \equiv \begin{pmatrix} \frac{d}{dr} + \frac{\epsilon_+}{r} + \frac{m\alpha_+}{\kappa_D} & -\frac{\alpha_s}{\kappa_D} \frac{d}{dr} + \frac{\alpha_v}{r} \\ \frac{\alpha_s}{\kappa_D} \frac{d}{dr} - \frac{\alpha_v}{r} & \frac{d}{dr} - \frac{\epsilon_-}{r} - \frac{m\alpha_-}{\kappa_D} \end{pmatrix}, \quad (23)$$

$\epsilon_{\pm} = \kappa_D + \alpha_s \alpha_{\pm} / \kappa_D$ and $\alpha_{\pm} = (\alpha_s \pm \alpha_v)$. The matrix operator L_D , obtained by means of SUSY QM, is reduced to the three-dimensional operator L reported in Ref. 7, which has been constructed by imposing an intertwining relation between the corresponding radial Dirac Hamiltonian and L .

In order to find the expression for the radial components of the SUSY ground state, we define

$$\begin{pmatrix} \tilde{G}_{0\kappa_D}^{(1)} \\ \tilde{G}_{0\kappa_D}^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\alpha_s}{\kappa_D} \\ \frac{\alpha_s}{\kappa_D} & 1 \end{pmatrix} \begin{pmatrix} G_{0\kappa_D}^{(1)} \\ G_{0\kappa_D}^{(2)} \end{pmatrix}. \quad (24)$$

Therefore, equation (23) is rewritten as

$$\left\{ \frac{d}{dr} + \frac{1}{r} \begin{pmatrix} \kappa_D & \alpha_+ \\ \alpha_- & -\kappa_D \end{pmatrix} \right\} \begin{pmatrix} \tilde{G}_{0\kappa_D}^{(1)} \\ \tilde{G}_{0\kappa_D}^{(2)} \end{pmatrix} = -\frac{m\kappa_D}{\kappa_D^2 + \alpha_s^2} \begin{pmatrix} \alpha_+ & \frac{\alpha_s \alpha_+}{\kappa_D} \\ \frac{\alpha_s \alpha_-}{\kappa_D} & -\alpha_- \end{pmatrix} \begin{pmatrix} \tilde{G}_{0\kappa_D}^{(1)} \\ \tilde{G}_{0\kappa_D}^{(2)} \end{pmatrix}, \quad (25)$$

which can be easily solved by diagonalizing the matrix of the factor $1/r$. For this purpose, we perform the transformation

$$\begin{pmatrix} \tilde{F}_{0\kappa_D}^{(1)} \\ \tilde{F}_{0\kappa_D}^{(2)} \end{pmatrix} = \begin{pmatrix} \kappa_D + \gamma & \alpha_+ \\ -\alpha_- & \kappa_D + \gamma \end{pmatrix} \begin{pmatrix} \tilde{G}_{0\kappa_D}^{(1)} \\ \tilde{G}_{0\kappa_D}^{(2)} \end{pmatrix}. \quad (26)$$

Thus, from equation (25), we obtain

$$\left\{ \frac{d}{dr} + \frac{1}{r} \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} \right\} \begin{pmatrix} \tilde{F}_{0\kappa_D}^{(1)} \\ \tilde{F}_{0\kappa_D}^{(2)} \end{pmatrix} = -\frac{m}{\kappa_D^2 + \alpha_s^2} \begin{pmatrix} \alpha_v \kappa_D + \alpha_s \gamma & 0 \\ 0 & \alpha_v \kappa_D - \alpha_s \gamma \end{pmatrix} \begin{pmatrix} \tilde{F}_{0\kappa_D}^{(1)} \\ \tilde{F}_{0\kappa_D}^{(2)} \end{pmatrix}. \quad (27)$$

The unnormalized solutions for these differential equations are given by

$$\tilde{F}_{0\kappa_D}^{(1)} = r^{-\gamma} \exp \left(-\frac{m}{\kappa_D^2 + \alpha_s^2} (\alpha_v \kappa_D + \alpha_s \gamma) \right), \quad (28)$$

$$\tilde{F}_{0\kappa_D}^{(2)} = r^{\gamma} \exp \left(-\frac{m}{\kappa_D^2 + \alpha_s^2} (\alpha_v \kappa_D - \alpha_s \gamma) \right). \quad (29)$$

Since $\tilde{F}_{0\kappa_D}^{(1)}$ diverges at $r = 0$, it is not a physically acceptable solution. Hence, the radial spinor for the supersymmetric ground state is

$$\psi_{SUSY}^0 \equiv \begin{pmatrix} 0 \\ r^\gamma \exp\left(-\frac{m}{\kappa_D^2 + \alpha_s^2}(\alpha_v \kappa_D - \alpha_s \gamma)\right) \end{pmatrix}. \quad (30)$$

For the case $\alpha_s = 0$, ψ_{SUSY}^0 is a normalizable solution only for $\kappa_D < 0$ which is in accordance to the results reported in Ref. 3 for $3 + 1$ dimensions.

Having obtained the SUSY ground state, the explicit form of the eigenfunctions corresponding to higher SUSY energy levels should be determined by solving the eigenvalue equation $\mathcal{H}\Psi_{SUSY} = b^2\Psi_{SUSY}$ which is equivalent to find the solutions of the equation $B^2\Psi_{\kappa_D} = b^2\Psi_{\kappa_D}$. Nevertheless, this problem is much more complicated to solve than the original Dirac eigenvalue equation.

IV. CONCLUDING REMARKS

We treated the Dirac-Kepler problem plus Coulomb-type scalar potential by generalizing the Lippmann-Johnson operator in general dimensions. This operator allowed us to construct the supersymmetric charges from which we found the ground state energy spectrum and the spectrum energy for discrete excited states. The action of the Lippmann-Johnson operator on the ground state of the Dirac equation leads to the radial operator L_D , equation (23), which generalizes to D -dimensions that reported in Ref. 7 for the three-dimensional space. The ground state obtained in this paper is in full agreement with those reported in Ref. 14 and it reduces to that in three dimensions reported in Ref. 6. With the generalized Lippmann-Johnson operator reported in the present paper, we can construct an $SO(4)$ symmetry treatment, similar to that given for the Kepler-Coulomb problem in three spatial dimensions¹⁶, which is a work in progress.

Appendix A: Calculation of $[B, H] = 0$

For an arbitrary radial function $f(r)$, we show that $[K_D, f(r)] = 0$. Moreover, we find that

$$[H, \gamma^0] = 2\gamma^i p^i, \quad (\text{A1})$$

$$[H, \gamma^{D+1}] = -2\gamma^{D+1}\gamma^0(m + V_s), \quad (\text{A2})$$

$$\left[H, \gamma^{D+1}\gamma^0\gamma^a\frac{x^a}{r} \right] = \frac{2i}{r}\gamma^{D+1}\gamma^0 K_D. \quad (\text{A3})$$

Hence,

$$\begin{aligned} [B, H] &= -iK_D [H, \gamma^{D+1}] (H - \gamma^0 m) \\ &\quad + imK_D \gamma^{D+1} [H, \gamma^0] \\ &\quad + \left[H, \gamma^{D+1}\gamma^0\gamma^a\frac{x^a}{r} \right] (\alpha_v m + \alpha_s H) \\ &= -2iK_D \gamma^{D+1}\gamma^0 \{ (m + V_s) (H - \gamma^0 m) \\ &\quad - m\gamma^0\gamma^a p^a - (V_v m + V_s H) \} \\ &= 0. \end{aligned} \quad (\text{A4})$$

Appendix B: Calculation of $B\Psi_{\kappa_D}$

Considering the properties of the operator K_D , the algebra satisfied by the matrices γ^i and from equations (9) and (10), we obtain

$$\begin{aligned} &-iK_D \gamma^{D+1} (H - \gamma^0 m) \Psi_{\kappa_D} \\ &= i \left\{ \gamma^{D+1}\gamma^0\gamma^a\frac{x^a}{r} \left[\frac{x^b}{r} p^b - \frac{i}{r} \left(\gamma^0 K_D + \frac{D-1}{2} \right) \right] \right. \\ &\quad \left. + (V_v - \gamma^0 V_s) \gamma^{D+1} \right\} K \Psi_{\kappa_D} \\ &= \kappa_D r^{-\frac{D-1}{2}} \left\{ -(\partial_r + \kappa_D \gamma^0) \Phi_{\kappa_D} \right. \\ &\quad \left. + (V_v - \gamma^0 V_s) \Theta_{\kappa_D} \right\} \end{aligned} \quad (\text{B1})$$

and

$$\begin{aligned}
& \gamma^{D+1} \gamma^0 \gamma^i \frac{x^i}{r} (\alpha_v m + \alpha_s H) \Psi_{\kappa_D} \\
&= -r^{-\frac{D-1}{2}} \left\{ \alpha_s \left(\partial_r - \frac{\kappa_D}{r} \gamma^0 \right) \Theta_{\kappa_D} \right. \\
&\quad \left. + [\alpha_s \gamma^0 (m + V_s + \gamma^0 V_v) + m \alpha_v] \Phi_{\kappa_D} \right\}. \tag{B2}
\end{aligned}$$

Thus, the explicit form of the Lippmann-Johnson operator B acting on an eigenstate Ψ_{κ_D} of the Hamiltonian H is

$$\begin{aligned}
B \Psi_{\kappa_D} = -r^{-\frac{D-1}{2}} \left\{ (\alpha_s \partial_r - \kappa_D V_v) \Theta_{\kappa_D} + \left(\kappa_D \left(\partial_r + \frac{\kappa_D}{r} \gamma^0 \right) + m \alpha_v \right) \Phi_{\kappa_D} \right. \\
\left. + \alpha_s \gamma^0 (m + V_s + \gamma^0 V_v) \Phi_{\kappa_D} \right\}. \tag{B3}
\end{aligned}$$

Appendix C: Calculation of B^2

With the definitions $A_1 = H - \gamma^0 m$ and $A_2 = \alpha_v m + \alpha_s H$, we find the following commutation relations

$$[A_1, A_2] = -2m\alpha_s \gamma^i p^i, \tag{C1}$$

$$[A_1, \gamma^{D+1}] = -2\gamma^{D+1} \gamma^0 V_s, \tag{C2}$$

$$[A_2, \gamma^{D+1}] = -2\alpha_s \gamma^{D+1} \gamma^0 (m + V_s), \tag{C3}$$

$$\left[A_1, \gamma^0 \gamma^i \frac{x^i}{r} \right] = \frac{2i\gamma^0}{r} K_D + 2V_s \gamma^i \frac{x^i}{r}, \tag{C4}$$

$$\left[A_2, \gamma^0 \gamma^i \frac{x^i}{r} \right] = 2iV_s \gamma^0 K_D + 2\alpha_s (m + V_s) \gamma^i \frac{x^i}{r}. \tag{C5}$$

Since $\left[K_D, \gamma^0 \gamma^i \frac{x^i}{r} \right] = 0$ and $\{K_D, \gamma^{D+1}\} = 0$, we finally obtain

$$\begin{aligned}
B^2 &= B^\dagger B = A_1^2 K_D^2 + A_2^2 - iA_1 \gamma^0 \gamma^i \frac{x^i}{r} A_2 K_D \\
&\quad + iA_2 \gamma^0 \gamma^i \frac{x^i}{r} A_1 K_D \\
&= A_1^2 K_D^2 + A_2^2 + 2m (V_v + \gamma^0 V_s) \gamma^0 K_D^2 \\
&= (H^2 - m^2) K_D^2 + A_2^2 \\
&= (\alpha_v m + \alpha_s H)^2 + K_D^2 (H^2 - m^2). \tag{C6}
\end{aligned}$$

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